ON THE MAGNETIC ENERGY OF SUPRACONDUCTORS

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1. A long stretched supraconductive body (wire), placed longitudinally in a homogeneous magnetic field of slowly increasing strength H_0 loses its supraconductivity when the field strength reaches a certain critical value $H_0 = H_c$. Measurements by de Haas and Voogd[1] disclosed that the case is different when the direction of the wire is transverse: then the transition takes place at a value of the applied field H_0 lower than H_c. By H_0 is meant the field as it would be if the supraconductor were absent. When it is present, the strength H is in general different from H_0 and varies from point to point, because the field cannot penetrate into the supraconductive material, so that the lines of induction are crowded at its surface. These conditions caused Von Laue[2] to enunciate the following hypothesis: *A supraconductive body begins its transition into the normal state when the strongest part of the magnetic field H at its surface reaches the critical value* $H = H_c$. Subsequent measurements by de Haas and co-workers on circular and elliptic cylinders[3] and on spheres[4] confirmed Von Laue's rule, so that it must be regarded as firmly established experimentally. Recently it has gained added theoretical importance, having been made by Landau[5] the starting point of interesting considerations on the structure of supraconductors during the process of transition.

The theoretical reasons for the dependence of the critical field H_c on the temperature were cleared up in a significant paper by Gorter and Casimir,[6] in which these authors also attempted a theoretical derivation of Von Laue's rule. However, this part of their work is, in our opinion, unconvincing and open to certain objections. In view of the great importance of the subject it seemed desirable to study it more closely in order to remove all doubts which may arise. The purpose of the present paper is to supply this want and to modify the derivation of Von Laue's rule in such a way as to meet the objections.

2. It is usually said that a supraconductive wire in a longitudinal homogeneous magnetic field does not appreciably change the field distribution.[7] The strength of field retains the value H_0 everywhere, except in the interior of the wire where it vanishes. When the wire (whose volume we denote by V) loses its supraconductivity, the increase of magnetic energy of the system is, therefore, $\Delta W_M = VH_0^2/8\pi$, i.e., equal to the field energy contained in its volume after the field has penetrated into it.

Let us now consider the general case of a supraconductor of any shape in an inhomogeneous field (Fig. 1); let its surface be initially *ABCDE;* and let the (tangential) field at the point C be denoted by H. Suppose that an infinitesimal volume element at the surface, $BCDFB$, loses its supraconductivity so that the volume of the body changes by the negative value ΔV. If it were possible to prove that the attendant increase of magnetic energy is

$$\Delta W_M = -\frac{H^2}{8\pi}\Delta V, \qquad (1)$$

then the same mathematical relations would obtain in the *general* conditions of any local supraconductive change as in the *special* conditions of the wire in a longitudinal field: if the law $H_0 = H_c$ obtains in the latter process, it should be $H = H_c$ in the former. In other words, the validity of the formula (1) implies the validity of Von Laue's rule.

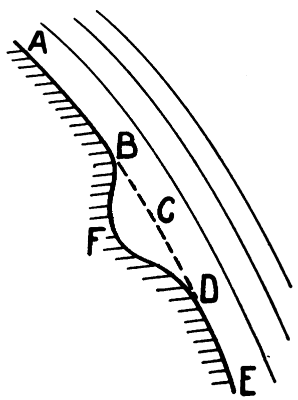

FIGURE 1

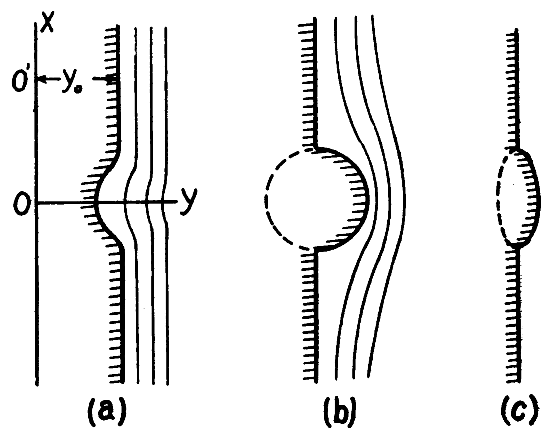

FIGURE 2

In their attempt to derive equation (1) Gorter and Casimir start from the case of a supraconductive half space with an indentation of infinitesimal volume in it. The other half space is filled with a homogeneous magnetic field, distorted at the indentation, and extending into infinity (Fig. 2a). We are unable to follow their argument nor can we confirm their results. We propose to show here (a) that the model of the infinite homogeneous field is inadequate for the purpose, giving a result three times too small; (b) that the validity of the formula (1) is not general.

If equation (1) is correct, it should apply also to positive ΔV, i.e., to humps as well as depressions. We may, therefore, begin with the case of a half spherical or a half ellipsoidal hump in a plane. Mathematically the problem is the same as that of a complete sphere or ellipsoid (dotted lines) in a homogeneous field, endless in all directions. It will be well to generalize the problem in two respects: (1) we shall consider a homogeneous body of any shape in an initially homogeneous field, (2) we attribute to it any magnetic permeability μ. If we characterize the interior and exterior of the body by the indices i and e (Fig. 3), the total field energy within a sphere of the large radius R is

$$W_M = \frac{1}{8\pi}\left[\mu \int H_i^2 d\tau + \int_e H_e^2 d\tau\right]. \qquad (2)$$

Since H can be derived from a potential, $H = -\nabla\Phi$, we find by means of Green's theorem

$$W_M = \frac{1}{8\pi}\left[\mu \int_{S_i} \Phi_i \frac{\partial \Phi_i}{\partial N} ds - \int_{S_i} \Phi_e \frac{\partial \Phi_e}{\partial N} ds + \int_{S_R} \Phi_e \frac{\partial \Phi_e}{\partial N} ds\right]. \qquad (3)$$

The conditions at the surface S_i are

$$\Phi_i = \Phi_e, \qquad \mu \frac{\partial \Phi_i}{\partial N} = \frac{\partial \Phi_c}{\partial N}.$$

Hence, the two first terms of (2) cancel out (in all cases, except $\mu = \infty$), and the energy becomes

$$W_M = \frac{1}{8\pi} \int_{S_R} \Phi_e \frac{\partial \Phi_e}{\partial R} ds. \qquad (4)$$

The potential Φ_e consists of two parts, $\Phi_e = \Phi_o + \Phi_p$, where Φ_o is the contribution of the original homogeneous field and Φ_p the perturbation due to the presence of the body. The expression for the first term is in polar coördinates $\Phi_o = -H_o r \cos \vartheta$, whereas Φ_p can be expanded into a series of spherical harmonics

$$\Phi_e = -H_o\left[r \cos\vartheta + A\frac{\cos\gamma}{r^2} + B\frac{K_2(\vartheta,\varphi)}{r^3} + \ldots\right], \qquad (5)$$

where ϑ denotes the angle between H_o and the radius vector, and γ between the direction of the dipole (ϑ_o, φ_o) and the radius vector, $\cos\gamma = \cos\vartheta_o \cos\vartheta + \sin\vartheta_o \sin\vartheta \cos(\varphi - \varphi_c)$.

Substituting into (3) we notice that, for $R \rightarrow \infty$, only the first and second term give a contribution to the integral, since the element of surface ds is proportional to R^2.

$$W_M = \frac{H_o^2}{8\pi}\left[\frac{4\pi}{3} R^3 + \frac{4\pi}{3} A_H\right], \qquad (6)$$

where $A_H = A \cos\vartheta_o$ is the component of the dipole in the direction of the field. The first term represents the energy in the absence of the body, W_o. In formula (1) the energy change is defined as $\Delta W_M = W_o - W_M$

i.e., the change that takes place when the body is removed. Since A_H is always proportional to the volume of the body, $4\pi A_H/3 = \alpha\Delta V$, we may write

$$\Delta W_M = (H_o^2/8\pi)\cdot(4\pi A_H/3), \tag{7}$$

$$\Delta W_M = -\alpha\frac{H_o^2}{8\pi}\Delta V, \tag{8}$$

where the coefficient α depends on the shape of the body.

The values of A_H and, consequently, of α may be found in any textbook of electricity. In the case of a sphere, $\alpha = (1-\mu)/(\mu + 2)$; for a very flat ellipsoid, $\alpha = (1-\mu)/3$. The last expression applies to any triaxial ellipsoid such that the axis oriented in the direction of the field is very large compared with one (or both) of the other axes. The particular conditions applying to supraconductors are obtained by putting $\mu = 0$, which gives for α in the two cases considered $^1/_2$ and $^1/_3$. As was mentioned above these results apply also to humps of corresponding shapes in a plane surface (Fig. 2b, c). In neither case does the calculated expression agree with formula (1).

To remove all doubts that the same conclusions apply also to very flat depressions as well as humps, we consider still one more case. Let us imagine a magnetic dipole placed in the point O of figure 2a, having the strength $A = -\kappa$ and direction OO'. This dipole will cause a distortion of the homogeneous field and bend the lines of force, in the plane AB into a shape indicated in the figure. We shall take the locus geometricus of the bent lines as the surface of the supraconductor and calculate the volume of the indentation and the energy due to it. The potential to the right of the surface AB is

$$\Phi = -H_o(r - \kappa/r^2)\cos\vartheta, \tag{9}$$

which is of the general form (5). However, the integral (4) must be extended only over the half space outside the body and not over the whole space. We have therefore to supply formula (7) with the factor $^1/_2$. Hence $\Delta W_M = (H_o^2/8\pi)\cdot(2\pi\kappa/3)$.

From the expression (9) of the potential it is not hard to calculate the equation of the surface AB. It turns out to have the form

$$y(1 + 2\kappa/r^3) = y_o.$$

The depression is shallow when κ/y_o^3 is small compared with 1. Making this assumption we neglect squares of this number and find for the change of volume due to the indentation $\Delta V = -2\pi\kappa$. This gives for ΔW_M the formula (8) with $\alpha = {}^1/_3$, just as in the case of the flat hump. We can therefore summarize our results for supraconductors as follows:

Semi-spherical hump $\alpha = {}^1/_2$
Flat hump or shallow depression with border line of any shape $\alpha = {}^1/_3$

3. The above results are, however, in contradiction with the method of calculating the magnetic energy change of a field from the magnetic moment produced by it. If the undisturbed field is appreciably homogeneous over the space later occupied by the body, the magnetic moment produced by it has a simple connection with the "strength of the dipole" A in formula (5), namely $M = H_oA$. The energy $-\Delta W_M$ of producing this dipole is according to the theory of electricity

$$-\Delta W_M = \int \mathbf{H}_o \cdot \mathbf{dM} = \frac{1}{2} H_o M \cos (M, H_o), \tag{10}$$

$$\Delta W_M = -\frac{H_o^2}{8\pi} 4\pi A_H, \tag{11}$$

or exactly three times as much as given by formula (6).

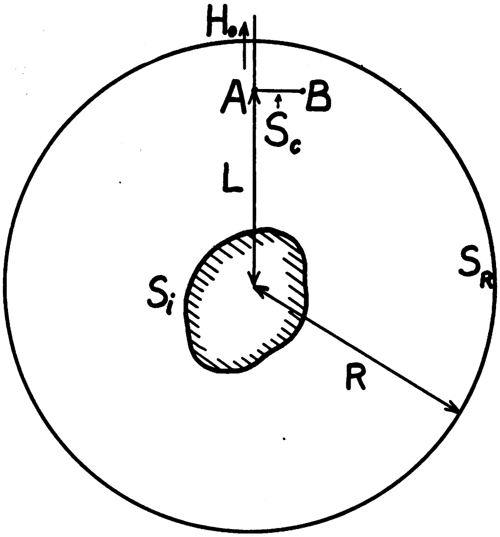

FIGURE 3

The discrepancy must be due to the inadequacy of the infinite homogeneous field. Such a field does not exist in nature and is only a *model*, an artificial device for making the problem definite. To resolve the contradiction we shall replace this model by one closer to reality. A magnetic field can be produced only by moving charges or *currents*. We shall prove that the *energy of a field originating in currents always obeys the formula* (11). We start from the simplest case of a closed linear current AB of the total current J and the very small plane area S_c, normal to the line AO connecting it with the body to be magnetized (Fig. 3). Since we wish the field produced by it to have the nearly constant value H_o over the extent of the body S_i we assume the distance L between A and O to be very large compared with the size of the body.

The potential of a linear current has at the point Q the expression $\Phi_c = J\omega_Q$, where ω_Q is the solid angle subtended by the current to the point

Q. This becomes in the very distant point $R = 0$, $\Phi_c = JS_c/L^3$, whence the strength of field $H_o = -\partial\Phi_c/\partial L = 2JS_c/L^3$. We find, therefore,

$$JS_c = \frac{1}{2}H_oL^3. \qquad (12)$$

Turning back to the transformation of the integrals (2) we notice that we can no longer apply Green's theorem to the space within the sphere of radius R (Fig. 3) because the potential Φ_c is many valued. To remove the many-valuedness we must lay a "branch cut" AB through the current (coinciding with what we called its *area* S_c). The space receives in this way two new border surfaces, the upper and lower sides of the cut AB, which we shall characterize by the second subscripts 2 and 1. The application of formula (3) is now permissible provided the integral is extended over the new borders. On the other hand, the integral over the infinite sphere vanishes, because the potential $\Phi = \Phi_c + \Phi_p$ decreases in infinity proportionally to $1/R^2$. We find, therefore, instead of equation (4)

$$W_M = -\frac{1}{8\pi}\int_{S_c}\left\{(\Phi_{c2} + \Phi_{p2})\left(\frac{\partial\Phi_{c2}}{\partial N} + \frac{\partial\Phi_{p2}}{\partial N}\right) - (\Phi_{c1} + \Phi_{p1})\left(\frac{\partial\Phi_{c1}}{\partial N} - \frac{\partial\Phi_{p1}}{\partial N}\right)\right\}ds.$$

The energy W_o in the absence of the body is obtained from this expression by putting $\Phi_p = 0$. Moreover, the potential Φ_p as well as both the strengths $\partial\Phi_c/\partial N$ and $\partial\Phi_p/\partial N$ are *not* many valued being the same for both subscripts. Therefore

$$\Delta W_M = W_o - W_M = \frac{1}{8\pi}\int_{S_c}(\Phi_{c2} - \Phi_{c1})\frac{\partial\Phi_p}{\partial L}\,ds. \qquad (13)$$

According to the above expression for Φ_c the difference $\Phi_{c2} - \Phi_{c1} = J(\omega_2 - \omega_1) = 4\pi J = 2\pi H_oL^3/S_c$. In view of the largeness of L only the second term of (5) gives an appreciable contribution to the factor $\partial\Phi_p/\partial L$, namely $-2H_oA\cos\gamma$. Since $\gamma = \vartheta_o$ is here the angle between the field H_o and the dipole, we have $A\cos\gamma = A_H$ and

$$\Delta W_M = \frac{H_o^2}{8\pi}4\pi A_H \qquad (14)$$

in complete agreement with the expression (11).

The generalization of this result for any distribution of currents is a trivial matter and we shall only briefly indicate here the necessary steps. The most important step is to show that the result (14) is also true when the plane of the tiny closed current is not normal to the line OA but has any orientation. According to the Ampere-Stokes theorem, any closed current can be represented as a superposition of infinitesimal closed currents; this permits the immediate generalization for any current sufficiently far away from the body. Moreover, the magnetic moments are additive (the magnetic moment of any body is the sum of the magnetic moments of its parts). Therefore, restriction to wide currents may be removed by mentally breaking up the magnetized body into infinitesimal units.

It goes without saying that, when the body S_i and the current form a closed system, any energy increase of the body takes place at the expense of the energy of the current: the magnetization created in the body reacts upon the current in such a way that the total field energy remains unchanged.[8] This obvious fact is, however, irrelevant for our computation because we are interested in *the work done by outer forces against the body* S_i. The current is for us an external agency used to produce the magnetization in S_i; it is, therefore, best to imagine J maintained constant by external energy supplies.

We recall that the purpose of our investigation is to find whether the formula (1) is valid. In this connection our result that the factor of equation (8) must be multiplied by 3 compared with the values given at the end of section 2. We find, therefore, for supraconductors

Spherical hump in plane	$\alpha = {}^3/_2$
Flat hump or shallow depression of any border line	$\alpha = 1$

We see that the contention expressed in formula (1) *has no general validity but applies only to shallow depressions* (or flat humps). This is, however, quite sufficient to explain Von Laue's rule in supraconductive transitions: since the regions where the state is normal gradually penetrate into the supraconductor from its surface they must be necessarily shallow in the beginning of the process.

[1] de Haas and Voogd, *Comm. Leiden*, **214c** (1931).

[2] M. Von Laue, *Phys. Zeits.*, **33**, 793 (1932).

[3] de Haas, Voogd and Jonker, *Physica*, **1**, 280 (1934); de Haas and Casimir-Jonker, *Physica*, **1**, 291 (1934).

[4] de Haas and Guineau, *Physica*, **3**, 182, 534 (1935).

[5] L. Landau, *Phys. Zeit. Sowjetunion*, **11**, 129 (1937).

[6] Gorter and Casimir, *Physica*, **1**, 306 (1934).

[7] The correctness of this assertion follows from the discussion in section 3.

[8] Compare: H. A. Lorentz, *Marx's Handbuch der Radiologie*, Vol. VI, p. 141, Leipzig 1925; also: P. S. Epstein, *Proc. Nat. Acad. Sci.*, **12**, 634 (1926). The magnetization induced in the body S_i is also due to molecular or macroscopic currents, and its magnetic moment can be replaced by an induced closed current J_p. When this is done, two new terms in ΔW_M arise. (A) An integral of the same form as (13), but with the subscripts c and p interchanged, extended over S_p. This integral represents the mutual energy of the field and of the magnetic moment induced by it. It is identical with the expression (10). Both this term and (13) take then a form symmetrical in c and p, so that their numerical equality becomes directly evident. (B) The second arising term is the energy of the induced current itself, which must be oppositely equal to the first term, because of conservation of energy.